Stark Lifetimes for the Hydrogen Atom*

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The radiative transition probabilities in hydrogen have been calculated for all Stark transitions through $n=25$. In general, the total transition probability and lifetime of a particular state are a function of the parabolic quantum numbers n_1, n_2, and m. For $m>0$ and to within an error of the order of one percent the radiative lifetimes are functions only of n, m. The radiative lifetimes are presented graphically for all states from $n=2$ through $n=25$.

I. INTRODUCTION

In many processes involving the hydrogen atom the contributions from excited states become important, and it is frequently of interest to know the radiative lifetimes of these states. In most laboratory experiments involving energetic atoms, the atom is rarely in an environment free of external fields, and the radiative attenuation of the excited levels of interest is often determined by the Stark lifetimes.

The calculations of the radiative transition probabilities for many of the excited states in hydrogen have already been reported in the literature. The starting point for all such calculations is the general expression for the dipole-matrix element first derived in closed form by Gordon,[1] both in spherical coordinates (field-free case) and parabolic coordinates (Stark case). These expressions are generally quite complex and do not lend themselves easily to numerical evaluation. With the availability of modern computers, however, it has become feasible to extend these early calculations of the transition probability to very high atomic levels. In a recent paper, Green, Rush, and Chandler[2] have tabulated the squares of the dipole-matrix elements for the field-free $(n,1)$ states up to $n=60$. The reader is referred to their paper for a discussion of the early literature.

Relatively few calculations of the dipole-matrix elements between parabolic states are available. The squares of the matrix elements and the transition probabilities have been calculated for the $n=2$, 3 levels,[3,4] and for some of the states of the $n=5$ level.[5] More recently, Underhill and Waddell[6] have tabulated the oscillator strengths for all series members up to $n=18$ for the Lyman, Balmer, Paschen, and Brackett series in hydrogen. These tabulations are not sufficient, however, for determining the Stark lifetime beyond $n=5$. In this paper we discuss the results of calculations of the transition probabilities for all states through $n=25$. The tabulation of these results is too extensive to be listed here, but has been given elsewhere[7]; the radiative lifetimes of all the Stark states through $n=25$ are presented here graphically.

II. QUANTITATIVE DISCUSSION

The radiative transition probability coupling two states belonging to levels n, n' is given by

$$A(n';n_1'n_2'm'|n;n_1n_2m)=\frac{4}{3}\frac{e^2w^3a_0^2}{\hbar c^3}|(n';n_1'n_2'm'|\bar{r}|nn_1n_2m)|^2, \quad (1)$$

where n_1, n_2, and m are the parabolic quantum numbers related to the principal quantum number n by $n=n_1+n_2+m+1$.

In the evaluation of the matrix elements, we have used the relation

$$\begin{aligned}|(n';n_1'n_2'm'|\bar{r}|n;n_1n_2m)|^2 &= |(n';n_1'n_2'm|z|n;n_1n_2m)|^2 \\ &\quad +2|(n';n_1'n_2'm-1|x|n;n_1n_2m)|^2 \\ &\quad +2(1+\delta mo)|(n;n_1'n_2'm+1|x|n;n_1n_2m)|^2, \quad (2)\end{aligned}$$

where the expressions for the individual terms are given by Gordon[1]; in the expression for the first term on the right we have included a numerical correction factor in Gordon's original result which has been noted by Underhill.[5]

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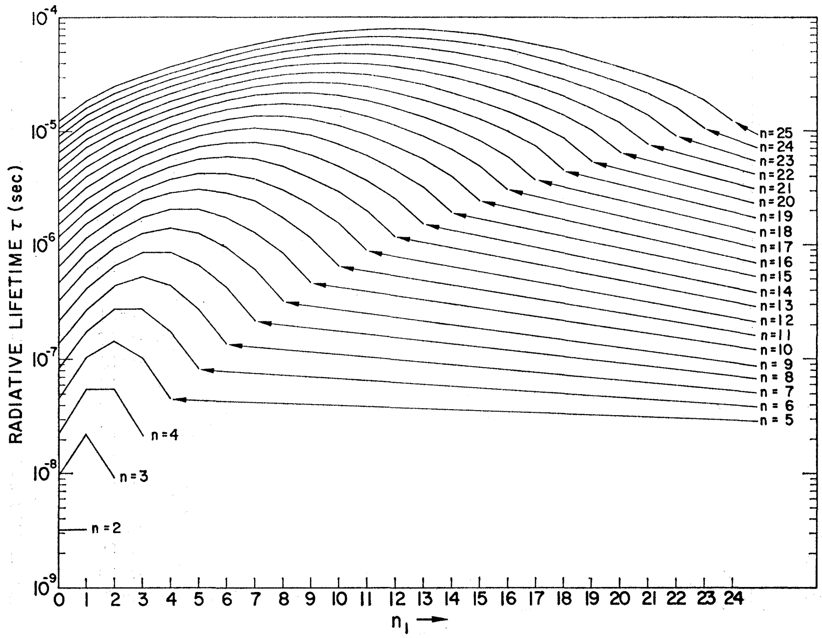

FIG. 1. Stark lifetimes ($m=0$).

The selection rules for dipole transitions between parabolic states of the atom are less restricted than those for dipole transitions between the spherical states. Accordingly, it is necessary to consider a much larger number of individual transitions in a Stark field than is necessary for an atom free of external fields. These individual transition probabilities have been summed to give the total transition probability for a state $(n;n_1n_2m)$,

$$A(n;n_1n_2m) = \sum_{n'<n} \sum_{n_1'=0}^{n'-m'-1} \sum_{n_2'=0}^{n'-n_1'-m'-1} A(n';n_1'n_2'm'|n;n_1n_2m)\,, \tag{3}$$

which is equal to the reciprocal of the radiative lifetime of the state.

In general, the radiative lifetime is a function of all three quantum numbers n_1n_2m. In Fig. 1 we have plotted the radiative lifetimes for all states through $n=25$ for the case $m=0$; note that the lifetimes are invariant under interchange of n_1 and n_2. Upon examining the transition probabilities (3) for states with $m>0$, it was noted that to a good approximation they were functions only of n, m. In Fig. 2 these approximate lifetimes $\tau(n,m)$ are plotted as a function of m for all levels up to $n=25$.

The strong dependence of the lifetimes on n_1, n_2 in the $m=0$ case compared with the $m>0$ case can be qualitatively understood in terms of the selection rule

$$A(n';n_1'n_2'm'|n;n_1n_2m)=0\,, \tag{4}$$

if $n_1'=n_2'n_1=n_2$; $m'=m$. This selection rule is a consequence of the positive symmetry of these states under reflection through the $z=0$ plane. For the $m=0$ case, the primary contribution to the total transition probability is the transition to the ground state; this transition is the $m\rightarrow m$ transition given by the first term on the right of Eq. (2). In this case and for intermediate values of n_1 and n_2, the condition (4) is satisfied. For $m=1$ the transition to the ground state can occur through the second term in Eq. (2) for all values of n_1 and n_2, there being no analog to condition (4) for these transitions. Similar considerations hold for $m>1$ transitions.

The lifetimes illustrated in Fig. 2 are exact for the $m=n-1$ states, there being only one allowed transition, $m\rightarrow m-1$. The greatest error in approximating the lifetimes as functions only of n, m is committed in the $m=1$ case. The variations in the lifetimes with n_1 or n_2 for this $m=1$ case range from approximately 2% at $n=10$, to approximately 3% at $n=25$. For further details concerning these variations the reader is referred to the tabulations.[7,8]

As a basis for checking the accuracy of the transition probabilities, we have calculated the matrix elements and transition probabilities through $n=25$ in spherical coordinates (field-free case), again using Gordon's expressions. These calculations of the (n,l) matrix elements were found to agree to six places through $n=20$ with the tabulated values given by Green, Rush, and Chandler.[2]

The transition probabilities for the field-free states with $l=n-1$ can be compared directly with the transition probabilities for the Stark states with $m=n-1$. For these limiting states the wave functions are identical in spherical and parabolic coordinates, and the selection

[8] D. A. Moody, Atomic Weapons Research Establishment Report (unpublished).

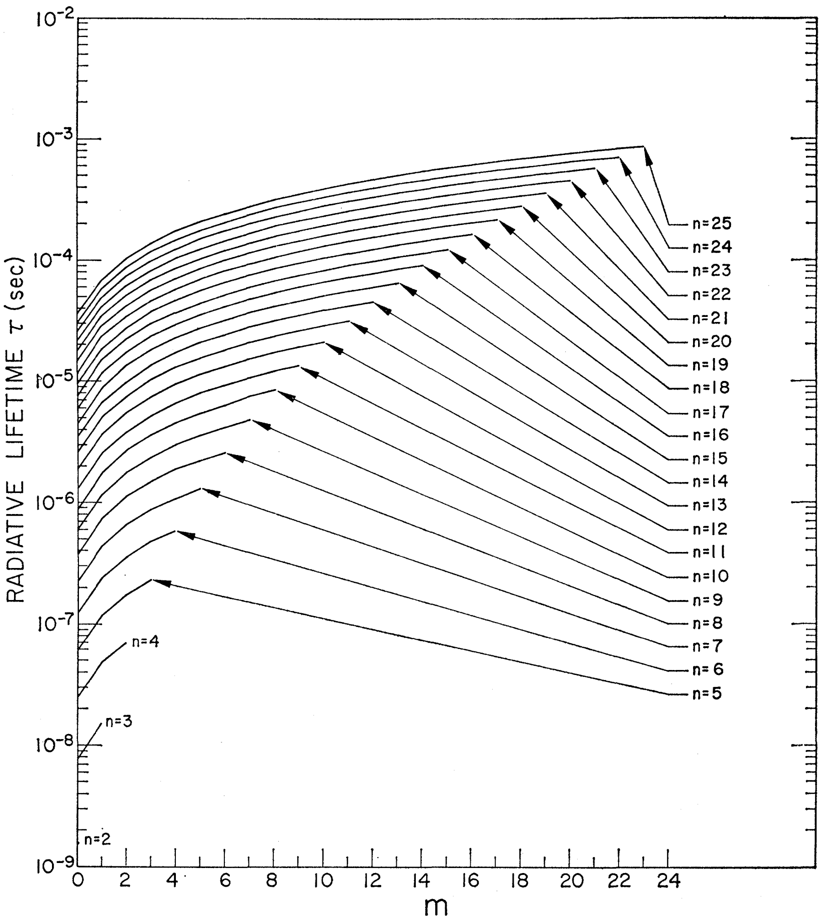

FIG. 2. Stark lifetimes ($m>0$).

rules $l \to l-1$, $m \to m-1$, respectively, allow only transitions to final states which are also identical in the two representations; it follows that the transition probabilities and the lifetimes are equal for these states.

From the dipole sum rule[4] we have the result that the sum of all transition probabilities coupling the states of any two levels is independent of the representation of the states, which can be expressed as

$$\sum_{l}(2l+1)A(nl)=\sum_{n_1}\sum_{m}A(n;n_1n_2m). \qquad (5)$$

We have compared these sums for the $n=2$, 3, 4, 5, 6, 9, 10, 14 levels; in all cases these sums are equal to five significant figures and precludes the possibility of any significant error in the lifetimes shown in the figures.

In Ref. 7 are tabulated the total transitions probabilities $A(n,l)$, and the individual transition probabilities $A(n';l'|n;l)$ for all spherical states through $n=25$. Also tabulated are the total Stark transition probabilities $A(n;n_1n_2m)$ for all states through $n=25$, and the individual Stark transition probabilities $A(n';n_1'n_2'm'|n;n_1n_2m)$ for all transitions through $n=10$. Similar tabulations of total and individual Stark transition probabilities are available in Ref. 8. This reference also includes tabulations in the same form as those of Ref. 6, of the oscillator strengths of all transitions in the range up to $n'=8$, $n=15$.

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